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CALCULATION OF RADIATIVE-CONDUCTIVE HEAT TRANSFER IN A SYSTEM CLOSED BY A SEMITRANSPARENT ENVELOPE

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A compact analytical solution of the one-dimensional problems of heat conduction is obtained for bodies of canonical shape with constant thermophysical properties and internal heat sources (sinks), whose power is generally coordinate-dependence. The solution includes, as partial cases, numerous known and new problems of stationary and nonstationary heat conduction, excluding the case of simultaneous assignment of second-second kind conditions on boundary surfaces.

In modern engineering, wide use is made of various devices, which can be treated as a closed or conventionally closed system bounded by an envelope permeable to radiation. The entire envelope or its separate parts can be permeable. In some cases, these envelopes can be made of semitransparent porous or perforated materials and are able not only to partially transmit radiation but also to ensure mass transfer. Systems of active heat protection, thermally loaded systems which include porous inserts, working sections of benches of radiation heating of structures, etc. are examples of these devices. In the general case, envelopes bounding these systems can be multilayer.

Radiative-conductive heat transfer in the indicated systems is usually calculated by the method of iterations, assuming the temperature state of all elements of the system at the initial instant of time to be known. To simplify the calculation of radiative heat transfer in the system, it is expedient to present the multilayer envelope as a conventional surface, the effective optical properties of which are identical to corresponding characteristics of the envelope [1]. A solution of the problem of radiative heat transfer in a multilayer transparent medium is presented, in particular, in [2], and a more detailed one is given in [3].

The characteristics of the radiation field for parts of the system that are found on the basis of methods expounded in these works allow calculation of radiative heat transfer in the considered region [4-6]. Refinement of the results obtained is made by the iteration process and is associated with determination of the temperature state of the bodies forming the system.

For the case where the envelope bounding the system and the bodies inside the system can have a canonical shape (plate, cylinder, sphere (solid or hollow), rod), one succeeds in obtaining a compact analytical solution of a one-dimensional nonstationary problem of heat conduction, which is convenient for an iteration calculation of radiative-conductive heat transfer in a closed system. We present the formulation of the corresponding boundary-value problem in dimensionless form as

$$\frac{\partial \theta}{\partial \mathrm{Fo}} = a_{\xi\xi} \frac{\partial^2 \theta}{\partial \xi^2} + b_{\xi} \frac{\partial \theta}{\partial \xi} + c\theta + F(\xi) , \tag{1}$$

$$\xi = \xi_1 : \ \alpha_1 \theta' (\xi_1) + \beta_1 \theta (\xi_1) = f_1 (\mathrm{Fo}) , \tag{2}$$

$$\xi = \xi_2 : \ \alpha_2 \theta' (\xi_2) + \beta_2 \theta (\xi_2) = f_2 (\mathrm{Fo}) , \tag{3}$$

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$$\text{Fo} = 0: \ \ \theta(\xi, 0) = f(\xi). \tag{4}$$

To improve the convergence of the solution of problem (1)-(4), it is expedient to determine the dimensionless temperature as the sum of nonstationary $\vartheta = \vartheta(\xi, \text{Fo})$ and quasistationary $\theta^* = \theta^*(\xi, \text{Fo})$ components [7]

$$\theta = \vartheta + \theta^*. \tag{5}$$

Substitution of (5) into (1)-(4) leads to the following two problems: for determination of a quasistationary (or stationary) component of the temperature field

$$a_{\xi\xi} \frac{\partial^2 \theta^*}{\partial \xi^2} + b_\xi \frac{\partial \theta^*}{\partial \xi} + c(\xi)\,\theta^* + F(\xi) = 0, \tag{6}$$

$$\alpha_1 \theta^{*'}(\xi_1) + \beta_1 \theta^*(\xi_1) = f_1(\text{Fo}), \tag{7}$$

$$\alpha_2 \theta^{*'}(\xi_2) + \beta_2 \theta^*(\xi_2) = f_2(\text{Fo}) \tag{8}$$

and for a nonstationary component of the temperature field

$$\frac{\partial \vartheta}{\partial \text{Fo}} = a_{\xi\xi} \frac{\partial^2 \vartheta}{\partial \xi^2} + b_\xi \frac{\partial \vartheta}{\partial \xi} + c(\xi)\,\vartheta + f(\theta^*), \tag{9}$$

$$\alpha_1 \vartheta'(\xi_1) + \beta_1 \vartheta(\xi_1) = 0, \tag{10}$$

$$\alpha_2 \vartheta'(\xi_2) + \beta_2 \vartheta(\xi_2) = 0, \tag{11}$$

$$\vartheta(\xi, 0) = f(\xi) - \theta^*(\xi, 0), \tag{12}$$

where

$$f(\theta^*) = -\frac{\partial \theta^*}{\partial \text{Fo}}.$$

The value of Fo in (7) and (8) and, in a more general case, also in the function $F(\xi) = F(\xi, \text{Fo})$ is used as a parameter [8].

A solution of differential equation (6) with boundary conditions (7) and (8) is obtained in the form

$$\theta^* = \left[\frac{b_2 b_3 - b_4}{b_1 b_3 - 1} - H_1(\xi)\right] \psi(\xi) + \left[\frac{b_1 b_4 - b_2}{b_1 b_3 - 1} + H_2(\xi)\right] \varphi(\xi), \tag{13}$$

where

$$b_1 = \left[\frac{\alpha_2 \psi'(\xi) + \beta_2 \psi(\xi)}{\alpha_2 \varphi'(\xi) + \beta_2 \varphi(\xi)}\right]_{\xi = \xi_2};$$

$$b_2 = \left[\frac{f_2 (\mathrm{Fo}) + [\alpha_2 \psi' (\xi) + \beta_2 \psi (\xi)] H_1 (\xi)}{\alpha_2 \varphi' (\xi) + \beta_2 \varphi (\xi)} - H_2 (\xi) \right]_{\xi=\xi_2} ;$$

$$b_3 = \left[\frac{\alpha_1 \varphi' (\xi) + \beta_1 \varphi (\xi)}{\alpha_1 \psi' (\xi) + \beta_1 \psi (\xi)} \right]_{\xi=\xi_1} ;$$

$$b_4 = \left[\frac{f_1 (\mathrm{Fo}) - [\alpha_1 \varphi' (\xi) + \beta_1 \varphi (\xi)] H_2 (\xi)}{\alpha_1 \psi' (\xi) + \beta_1 \psi (\xi)} + H_1 (\xi) \right]_{\xi=\xi_1} ;$$

$$H_1 (\xi) = \int \frac{F (\xi)}{a_{\xi\xi}} \frac{\varphi (\xi) \, d\xi}{\varphi (\xi) \psi' (\xi) - \psi (\xi) \varphi' (\xi)} ;$$

$$H_2 (\xi) = \int \frac{F (\xi)}{a_{\xi\xi}} \frac{\psi (\xi) \, d\xi}{\varphi (\xi) \psi' (\xi) - \psi (\xi) \varphi' (\xi)} .$$

The functions $\psi(\xi)$ and $\varphi(\xi)$ form a fundamental system of solutions of a second-order homogeneous differential equation (Eq. (6) at $F(\xi) = 0$). The form of these functions for some partial cases, which often occur in problems of heat conduction, is given in Table 1. In other cases these functions can be found from the literature by ordinary differential equations, e.g., [9].

For determination of a nonstationary component of temperature, it is convenient to use the method of finite integral transformations [8-12]. Taking, for this purpose, an integral transform of the form

$$\bar{\vartheta} (\mathrm{Fo}) = \int_{\xi_1}^{\xi_2} \rho (\xi) \, \vartheta (\xi, \mathrm{Fo}) \, \bar{k} (\lambda_n, \xi) \, d\xi$$

of Eq. (9) and initial condition (12), we obtain

$$\frac{d\bar{\vartheta} (\mathrm{Fo})}{d(\mathrm{Fo})} = - \lambda_n \bar{\vartheta} (\mathrm{Fo}) + \bar{f} (\theta^*) , \tag{14}$$

$$\bar{\vartheta} (0) = \bar{f} - \bar{\theta}^* (0) , \tag{15}$$

where

$$\bar{f} (\theta^*) = \int_{\xi_1}^{\xi_2} \rho (\xi) f (\theta^*) \, \bar{k} (\lambda_n, \xi) ; \quad \bar{f} = \int_{\xi_1}^{\xi_2} \rho (\xi) f (\xi) \, \bar{k} (\lambda_n, \xi) \, d\xi ;$$

$$\bar{\theta} (0) = \int_{\xi_1}^{\xi_2} \rho (\xi) \, \theta^* (\xi, 0) \, \bar{k} (\lambda_n, \xi) \, d\xi .$$

The weight function $\rho(\xi)$ is found from the formula

TABLE 1. Expressions for the Functions in Formula (13)

Shape of a body and coefficients in Eq. (6)	Homogeneous equation of heat conduction	Function $\psi(\xi)$	Function $\varphi(\xi)$
Plate: $a_{\xi\xi} = 1$; $b_{\xi} = c(\xi) = 0$	$\theta^* = 0$	ξ	1
Porous plate cooled by liquid: $a_{\xi\xi} = 1$; $b_{\xi} = \zeta_{\mathrm{w}} = \dfrac{Gc_{\mathrm{liq}} l}{\lambda_{\mathrm{w}}(1-\gamma)}$; $c(\xi) = 0$	$\theta^{*''} - \zeta_{\mathrm{w}}\theta^{*'} = 0$	$\exp(\zeta_{\mathrm{w}}\xi)$	1
Cylinder (solid or hollow): $a_{\xi\xi} = 1$; $b_{\xi} = 1/\xi$; $c(\xi) = 0$	$\theta^{*''} + \dfrac{1}{\xi}\theta^{*'} = 0$	$\ln \xi$	1
Sphere (solid or hollow): $a_{\xi\xi} = 1$; $b_{\xi} = 2/\xi$; $c(\xi) = 0$	$\theta^{*''} + \dfrac{2}{\xi}\theta^{*'} = 0$	$\dfrac{1}{\xi}$	1
Straight fin of a variable cross section – general case: $a_{\xi\xi} = \dfrac{S(\xi)}{S_0}$; $b_{\xi} = \dfrac{dS(\xi)}{d\xi}\dfrac{1}{S_0}$; $c(\xi) = -\mathrm{Bi}\dfrac{d\Phi(\xi)}{d(\xi)}\dfrac{1}{S_0}$	$a_{\xi\xi}\theta^{*''} + b_{\xi}\theta^{*'} + c(\xi)\theta^* = 0$	$\psi(\xi)$	$\varphi(\xi)$
Fin (rod) of a constant cross section: $a_{\xi\xi} = 1$; $b_{\xi} = 0$; $c(\xi) = -(ml)^2$; $m = \sqrt{\alpha\Pi/\lambda_{\mathrm{m}} S_0}$	$\theta^{*''} - (ml)^2\theta^* = 0$	$\exp(-\mathrm{ml}\,\xi)$	$\exp(ml\xi)$
Fin of triangular or trapezoidal cross section with a small apex angle: $a_{\xi\xi} = 1$; $b_{\xi} = 1$; $c(\xi) = -(ml)^2$; $m = \sqrt{\alpha/\lambda_{\mathrm{m}}\delta}$	$\xi\theta^{*''} + \theta^{*'} - (ml)^2\theta^* = 0$	$I_0(2ml\sqrt{\xi})$	$K_0(2ml\sqrt{\xi})$
Round fin of constant thickness equal to 2δ: $a_{\xi\xi} = 1$; $b_{\xi} = 1/\xi$; $c(\xi) = -(ml)^2$; $m = \sqrt{\alpha/\lambda_{\mathrm{m}}\delta}$	$\theta^{*''} + \dfrac{1}{\xi}\theta^{*'} - (ml)^2\theta^* = 0$	$I_0(ml\xi)$	$K_0(ml\xi)$

$$\rho = \rho(\xi) = \exp\left[-\int^{\xi} \frac{1}{a_{\xi\xi}}(a'_{\xi\xi} - b_{\xi})\, d\xi\right].$$

In the latter expression, any number within the range of from ξ_1 to ξ_2 is used as the lower limit of integration.

The kernel of the integral transform is found from the solution of the Sturm–Liouville boundary-value problem

$$-(pk')' + (q - \lambda^2\rho)\, k = 0\,, \tag{16}$$

$$\alpha_1 k'(\xi_1) + \beta_1 k(\xi_1) = 0\,, \tag{17}$$

TABLE 2. Expressions for the Functions in Formula (19)

Shape of a body and coefficients in Eq. (16)	Equation for the kernel of transformation $K(\lambda, \xi)$	Function $\psi(\lambda, \xi)$	Function $\varphi(\lambda, \xi)$	Note
Plate (monolith or porous): $\rho = 1$; $p = 1$; $q = 0$	$k'' + \lambda^2 k = 0$	$\cos(\lambda\xi)$	$\sin(\lambda\xi)$	
Cylinder (solid or hollow): $\rho = \xi$; $p = \xi$; $q = 0$	$k'' + \frac{1}{\xi}k' + \lambda^2 k = 0$	$J_0(\lambda\xi)$	$Y_0(\lambda\xi)$	
Sphere (solid or hollow): $\rho = \xi^2$; $p = \xi^2$; $q = 0$	$k'' + \frac{2}{\xi}k' + \lambda^2 k = 0$	$\xi^{-0.5}J_{0.5}(\lambda\xi)$	$\xi^{-0.5}Y_{0.5}(\lambda\xi)$	
Fin (rod) of a constant cross section: $\rho = 1$; $p = 1$; $q = (ml)^2$; $\omega^2 = \lambda^2 - (ml)^2$	$k'' + \omega^2 k = 0$	$\cos(\omega\xi)$ $\cosh(\omega\xi)$	$\sin(\omega\xi)$ $\sinh(\omega\xi)$	$\omega^2 > 0$ $\omega^2 < 0$
Fin of triangular or trapezoidal cross section with a small apex angle: $\rho = \xi$; $q = (ml)^2$; $\omega^2 = \lambda^2 - (ml)^2$	$\xi k'' + k' + \omega^2 k = 0$	$J_0(2\omega\sqrt{\xi})$ $I_0(2\omega\sqrt{\xi})$	$Y_0(2\omega\sqrt{\xi})$ $K_0(2\omega\sqrt{\xi})$	$\omega^2 > 0$ $\omega^2 < 0$
Round fin of constant thickness: $\rho = \xi$; $p = \xi$; $q = \xi(ml)^2$; $\omega^2 = \lambda^2 - (ml)^2$	$k'' + \frac{1}{\xi}k' + \omega^2 k = 0$	$J_0(\omega\xi)$ $I_0(\omega\xi)$	$Y_0(\omega\xi)$ $K_0(\omega\xi)$	$\omega^2 > 0$ $\omega^2 < 0$

$$\alpha_2 k'(\xi_2) + \beta_2 k(\xi_2) = 0, \tag{18}$$

where $p = a_{\xi\xi}\rho$; $q = -c(\xi)\rho$.

We present the solution of Eq. (16) in the form

$$k = k(\lambda, \xi) = \bar{B}_1 \psi(\lambda, \xi) + \bar{B}_2 \varphi(\lambda, \xi), \tag{19}$$

where $\psi(\lambda, \xi)$ and $\varphi(\lambda, \xi)$ is the fundamental system of solutions of Eq. (16) (Table 2).

Expression (18) contains three unknown quantities $\bar{B}_1$, $\bar{B}_2$, and λ, for determination of which we have only two boundary conditions (17) and (18). To eliminate this difficulty, both sides of (19) can be divided by one of the constants $\bar{B}_1$ or $\bar{B}_2$ and the unnormalized kernel of the integral transform $k(\lambda, \xi)$ can be determined accurate to an integration constant. This is based on the fact that in the solution of the problem considered only a normalized kernel is universally used for the nonstationary component of the temperature field and the normalization procedure allows elimination of the dependence of $\bar{k}(\lambda, \xi)$ on the accuracy of determination of $k(\lambda, \xi)$

$$\bar{k}(\lambda, \xi) = \frac{k(\lambda, \xi)}{\sqrt{N}}, \tag{20}$$

where the norm is found from the expression

$$N = \int_{\xi_1}^{\xi_2} \rho(\xi)\, k^2(\lambda, \xi)\, d\xi. \tag{21}$$

Having divided both sides of Eq. (19) by $\bar{B}_1$ and having used conditions (17) and (18), we find a new value of B_2:

$$B_2 = \frac{\bar{B}_2}{\bar{B}_1} = -\frac{\alpha_1 \psi'(\lambda, \xi_1) + \beta_1 \psi(\lambda, \xi_1)}{\alpha_1 \varphi'(\lambda, \xi_1) + \beta_1 \varphi(\lambda, \xi_1)}, \tag{22}$$

and the characteristic equation for determining eigenvalues

$$\alpha_2\varphi'(\lambda_n, \xi_2) + \beta_2\varphi(\lambda_n, \xi_2) = -\frac{1}{B_2}[\beta_2\psi(\lambda_n, \xi_2) + \alpha_2\psi'(\lambda_n, \xi_2)] . \tag{23}$$

Division of (19) by $\bar{B}_1$ instead of $\bar{B}_2$ gives a new value $B_1 = 1/B_2$, and expression (23) remains constant.

Thus, the relation for determination of the unnormalized kernel acquires one of the following forms:

$$k(\lambda_n, \xi) = \psi(\lambda_n, \xi) + B_2\varphi(\lambda_n, \xi)$$

or

$$k(\lambda_n, \xi) = B_1\psi(\lambda_n, \xi) + \varphi(\lambda_n, \xi) ,$$

where B_2 is found from (2) and $B_1 = 1/B_2$.

The functions $\psi(\lambda_n, \xi)$ and $\varphi(\lambda_n, \xi)$, which form the fundamental system of solutions of some widely encountered equations (partial cases of Eq. (16)) that are used for determining the kernels of integral transform $\bar{k}(\lambda_n, \xi)$ in problems of heat conduction, are given in Table 2. The normalized kernel is found from expression (20) using (21). The expressions for the kernels of a finite transform and characteristic equations for determination of eigenvalues as applied to partial problems of heat conduction with specific boundary conditions are given in tabulated form in [11, 12]. A solution for the transform of the nonstationary component of the temperature field, which satisfies Eq. (14) and initial condition (15), is obtained in the form

$$\bar{\vartheta}(\mathrm{Fo}) = \exp(-\lambda_n^2\,\mathrm{Fo})\left[\bar{f} - \bar{\theta}^*(0) + \int_0^{\mathrm{Fo}} \bar{f}(\theta^*)\exp(\lambda_n^2\,\mathrm{Fo})\,d\mathrm{Fo}\right].$$

Using the inversion formula [10], we find

$$\vartheta(\xi, \mathrm{Fo}) = \sum_{n=1}^{\infty} \bar{\vartheta}(\mathrm{Fo})\,\bar{k}(\lambda_n, \xi) . \tag{24}$$

A solution of the initial problem (1)-(4) results from the summation of the solutions of quasistationary and nonstationary components of the temperature field

$$\theta = \theta(\xi, \mathrm{Fo}) = \theta^*(\xi) + \vartheta(\xi, \mathrm{Fo}) , \tag{25}$$

where the functions on the right-hand side of (25) are calculated by formulas (13) and (24), respectively.

The solution obtained contains, in a generalized form, a large class of known [13] and new problems of stationary (13) and nonstationary (25) heat conduction for bodies of a canonical shape with constant thermophysical properties and an assigned law of distribution of the internal heat sources (sinks). The solution is particularly convenient when in the considered system of bodies, there are bodies of various geometry simultaneously.

NOTATION

θ, dimensionless temperature; θ^*, dimensionless quasistationary or stationary component of temperature; ϑ, dimensionless nonstationary component of temperature; ξ, dimensionless coordinate; $c(\xi)$, b_ξ, and $a_{\xi\xi}$, coefficients in the heat conduction equation at the temperature and its first and second derivatives, respectively; $F(\xi)$, known function of the coordinate in the heat conduction equation; α_1, α_2, β_1, and β_2, coefficients in generalized representation of boundary conditions; $f_1(\mathrm{Fo})$ and $f_2(\mathrm{Fo})$, known functions of time or constants in boundary conditions; Fo and Bi, Fourier and Biot numbers; $k = k(\lambda_n, \xi)$ and $\bar{k}(\lambda_n, \xi)$, unnormalized and normalized kernels of integral transform; λ_n, eigenvalues; $S(\xi)$ and S_0, current value of the cross-sectional area and the base of a fin, respectively; $\varphi(\xi)$, current value of the surface of convective heat transfer of a fin; l, fin height or wall thickness; α and λ_m, coefficients of heat transfer from the surface of a fin and thermal conductivity of its material; Π, perimeter of a fin of constant cross section; δ, half-thickness of a fin; G, specific mass flow rate of liquid; c_{liq}, heat capacity of liquid; λ_w and γ, thermal conductivity of a porous wall and its porosity, respectively.

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